Lecture 12: Stability Analysis

12.1 Motivation

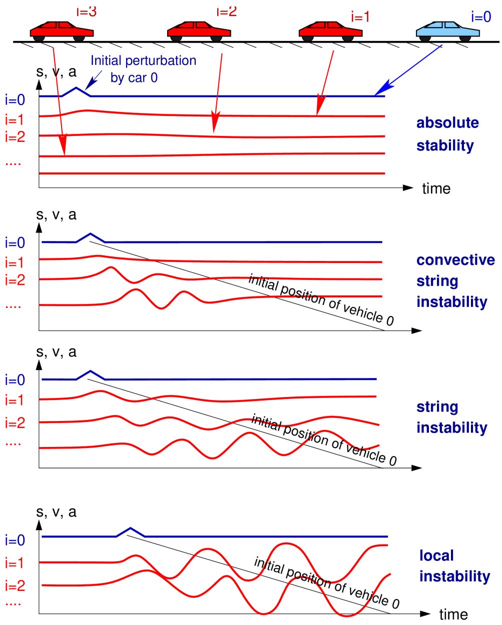
- At time $t = t_0$, the driver of car 1 brakes slightly (for whatever reason)
- As a result, the new optimal speed for car 2 is given by v_1 as well. So the driver of this car reduces the speed to v_1 at time t_1
- If traffic is sufficiently **dense** and/or the **speed adaptation time** is sufficiently long and/or $|V_e'(s)|$ **large**, the gap $s_2(t_1) < s_e(v_1) \Rightarrow$ further deceleration to $v_2 < v_1$
- Car 3 approaches to a gap smaller than $s_e(v_2) \Rightarrow$ braking to $v_3 < v_2$
- The vicious circle eventually leads to full stops

12.2 Mathematical Classification

Instabilities can be classified according to

- Evolution over time: **Absolutely stable**, **convectively string unstable**, (absolutely) **string unstable**, **locally unstable**
- Type of perturbation and endpoint: Small temporary perturbations remain small: **Ljapunov stability**; small temporary perturbation tend to zero: **asymptotic stability**; small *persistent* perturbations do not significantly change the system: *structural stability*
  For continuous many-vehicle microscopic or macroscopic systems, all three concepts are equivalent to string stability which therefore can be investigated with temporary extended perturbations (wave ansatz) or with permanent local perturbations (Laplace approach)
- Amplitude of perturbation: **linear** vs **nonlinear stability**
- Instability of system vs numerical integration code: **system** vs *numerical* instability

Evolution over time



Perturbation response $u_j(t)$ of follower j [take index j because i will be the imaginary unit, later on] at time t to an temporary perturbation of leader $j=0$ at $x=0$ around time 0:

- (upstream) Convective string *instability* for an infinite platoon: string *unstable* but

$$\lim_{t\to\infty} u_j(t) = 0, \text{ if } x(t) \le 0$$

- String stability of an infinite platoon

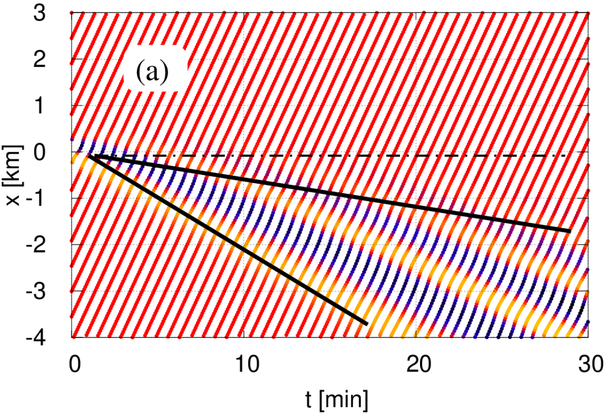
$$\lim_{t\to\infty} \max_j (u_j(t)) = 0.$$

- Local stability:

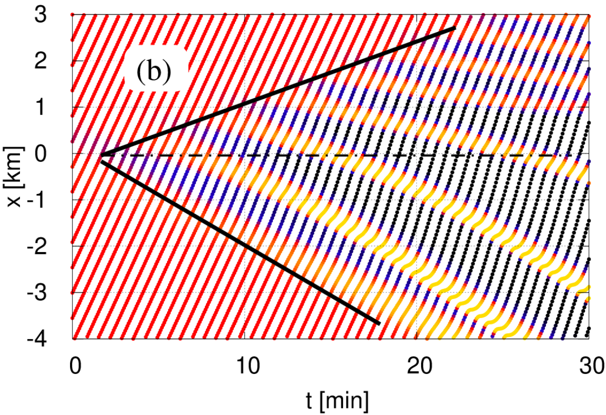
$$\lim_{t\to\infty} u_j(t) = 0 \text{ for all finite } j.$$

Convective instability

Upstream convective instability



Absolute convective instability



12.3 Local Stability: Analysis

Ansatz: perturbations from the steady-state $(s_e(v_e), v_e)$ of a follower $j = 1$ following a leader driving at constant speed v_e

$$\begin{aligned} s_1(t) &= s_e + y(t), \\ v_1(t) &= v_e + u(t). \end{aligned}$$

Insert into a general car-following model $\dot{v_j} = f(s_j, v_j, v_{j-1}) \equiv f(s, v, v_l)$ and linearize:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = u_l - u = -u, \tag{1}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f_s y + f_v u + f_l u_l = f_s y + f_v u \tag{2}$$

where (Taylor expansion of $f(.)$ to first order)

$$f(s, v, v_l) = f(s_e, v_e, v_e) + f_s y + f_v u + f_l u_l + \text{ higher orders}$$

- Steady state implies $f(s_e, v_e, v_e) = 0$
- Taylor coefficients are partial derivatives (*acceleration sensitivities*)

$$f_s = \left.\frac{\partial f}{\partial s}\right|_e, \quad f_v = \left.\frac{\partial f}{\partial v}\right|_e, \quad f_l = \left.\frac{\partial f}{\partial v_l}\right|_e$$

Local Stability: Results

Taking the time derivative of (1) and insert (2): $\ddot{y} = -\dot{u} = -f_s y - f_v u = -f_s y + f_v \dot{y}$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - f_v \frac{\mathrm{d}y}{\mathrm{d}t} + f_s y = 0$$

Write this as an harmonic oscillator:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\eta \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \omega_0^2 y(t) = 0, \quad \eta = -\frac{f_v}{2}, \quad \omega_0^2 = f_s$$

Ansatz $y(t) = e^{\lambda t}$ gives, *for any model*, the local dynamics in terms of just two sensitivities f_s and f_v:

$$\boxed{\lambda_{1/2} = -\eta \pm \sqrt{\eta^2 - \omega_0^2} = \frac{f_v}{2} \pm \sqrt{\frac{f_v^2}{4} - f_s}}$$

- Sufficient condition for local stability: $f_v < 0$ AND $f_s \geq 0$: always satisfied if the plausibility criteria are met
- Overdamped return to the steady state (no oscillations) if $\mathrm{Im}(\lambda) = 0$ or $f_s \leq \frac{f_v^2}{4}$
- Sensitivity f_l or $(\tilde{f}_v)$ drops out since leader drives at constant speed

12.4 String Stability of Car-Following Models: Two approaches

Criterion	Wave ansatz	Laplace ansatz
System	infinite or closed	semi-infinite platoon
Perturbations, time	temporary	permanent
Perturbations, space	extended	localized to a single leader
System boundaries	initial conditions	boundary conditions for leader
Definition string instability	temporally growing perturbations	perturbations growing from follower from follower
Advantages	analysis of convective instability, extendable to macroscopic models	analytic identification of the most unstable mode, inclusion of lower-level control/delays, analysis of heterogeneous traffic

Laplace analysis (Simpler than the wave approach and sufficient here)

Start as in the local approach with linearized perturbations $s = s_e + y$, $v = v_e + u$ of the CF model $f(s, v, v_l) \Rightarrow$ but this time including the response to the leader and retaining the vehicle indices j and $j-1$ (we need i for the imaginary unit $i = \sqrt{-1}$):

$$\begin{aligned}\dot{y}_j &= u_{j-1} - u_j, \\ \dot{u}_j &= f_s y_j + f_v u_j + f_l u_{j-1}\end{aligned}$$

Coupled equations for the speed deviations alone:

$$\ddot{u}_j = f_s(u_{j-1} - u_j) + f_v \dot{u}_j + f_l \dot{u}_{l-1}$$

Laplace Ansatz $\boxed{u_j(t) = \hat{u}_j e^{\lambda t} = \hat{u}_j e^{i\omega t}}$ (Why λ is purely imaginary?) Laplace perturbations are stationary and only increase from vehicle to vehicle gives complex **transfer function**

$$\boxed{G(\mathrm{i}\omega) = \frac{\hat{u}_j}{\hat{u}_{j-1}} = \frac{\hat{y}_j}{\hat{y}_{j-1}} = \frac{\lambda f_l + f_s}{\lambda^2 - \lambda f_v + f_s} = \frac{\mathrm{i}\omega f_l + f_s}{-\omega^2 - \mathrm{i}\omega f_v + f_s}}$$

For each harmonic component of the leader's oscillation, the next follower responds

- with a phase shift $\arctan\left[\mathrm{Im}(G(\mathrm{i}\omega))/\mathrm{Re}(G(\mathrm{i}\omega))\right]$,
- and a growth factor $|G(\mathrm{i}\omega)|$ with $|G(0)| = G(0) = 1$ Why? $\omega = 0$ reflects steady state

Instability and wavelength of most unstable mode

Squared absolute growth factor is a function of ω^2:

$$|G(\mathrm{i}\omega)|^2 = \frac{f_s^2 + f_l^2\omega^2}{(f_s - \omega^2)^2 + f_v^2\omega^2} := G_{\text{abs}}^2(\omega^2) \tag{3}$$

Necessary condition for the *resonance* frequency of the most unstable (most growing) mode (lengthy calculation):

$$\frac{\mathrm{d}G_{\text{abs}}^2}{\mathrm{d}\omega^2} \stackrel{!}{=} 0 \quad \Rightarrow \quad \omega_{\text{res}}^2 = \frac{f_s}{f_l^2}\left(-f_s + \sqrt{f_s^2 + f_l^2(f_l^2 - f_v^2 + 2f_s)}\right)$$

A maximum for real-valued, positive ω_{res} with $|G(\omega_{\text{res}})| > 1$ Why > 1? because $G(0) = 1$ and ω_{res} denotes the *maximum* only exists if

$$\boxed{f_l^2 - f_v^2 + 2f_s > 0 \quad \text{string instability criterion}}$$

- String instability is equivalent to at least some oscillations increasing from car to car
- Both the growth factor and the resonance frequency of the fastest growing mode increase strictly monotonously with the string instability indicator $f_l^2 - f_v^2 + 2f_s$.
- At neutral stability, the resonance frequency ω_{res} of the maximum growth tends to zero $\Rightarrow$ **Long-wavelength instability**

Different forms of the string instability criterion

The acceleration function of some models is given by $\tilde{f}(s, v, \Delta v)$ instead of $f(s, v, v_l)$.
Conversion:

$$f_s = \tilde{f}_s, \quad f_v = \tilde{f}_v + \tilde{f}_{\Delta v}, \quad f_l = -\tilde{f}_{\Delta v}$$

Often, the condition gets simpler when expressing the gap sensitivity in terms of the gradient of the speed-gap relation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(f(s, v_e(s), v_e(s))\right)_e = f_s + f_v v_e'(s) + f_l v_e'(s) \quad \Rightarrow \quad v_e'(s) = -\frac{-f_s}{f_v + f_l},$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(\tilde{f}(s, v_e(s), 0)\right)_e = \tilde{f}_s + \tilde{f}_v v_e'(s) \quad \Rightarrow \quad v_e'(s) = -\frac{-\tilde{f}_s}{\tilde{f}_v}$$

$\Rightarrow$ Four forms of the string instability condition:

	Model $f(s, v, v_l)$	Model $\tilde{f}(s, v, \Delta v)$
With all three sensitivities	$2f_s - f_v^2 + f_l^2 > 0$	$2\tilde{f}_s - \tilde{f}_v^2 - 2\tilde{f}_v\tilde{f}_{\Delta v} > 0$
With speed-gap gradient	$v_e'(s_e) > (f_l - f_v)/2$	$v_e'(s_e) > -(\tilde{f}_v + \tilde{f}_{\Delta v})/2$

Model examples I: OVM/FVDM stability condition

- Acceleration equation:

$$\tilde{f}(s, v, \Delta v) = (v_e(s) - v)/\tau - \gamma \Delta v \quad \text{(with } \gamma = 0 \text{ for the FVDM)}$$

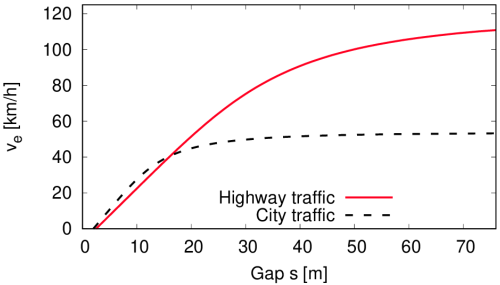
- Relevant sensitivities: $\tilde{f}_v = -\frac{1}{\tau}, \quad \tilde{f}_{\Delta v} = -\gamma$
- Instability criterion: $v_e'(s) > \frac{1}{2\tau} + \gamma$: increased stability for increased agility $1/\tau$ and increased anticipation γ

Model examples II: IDM instability condition

$$f^{\text{IDM}}(s, v, v_l) = a\left[1 - \left(\frac{v}{v_0}\right)^4 - \left(\frac{s^*}{s}\right)^2\right], \quad s^* = s_0 + vT + \frac{v(v - v_l)}{2\sqrt{ab}}$$

Because variations around the steady state are considered, the max condition for s^* is not needed here

$$\begin{aligned}
f_v^{\text{IDM,free}} &= -\frac{4a}{v_e}\left(\frac{v_e}{v_0}\right)^4, \\
f_v^{\text{IDM,int}} &= -\frac{(s_0 + v_e T)\left(2aT + \sqrt{\frac{a}{b}}v_e\right)}{s_e^2}, \\
f_l^{\text{IDM}} &= \sqrt{\frac{a}{b}}\left(\frac{(s_0 + v_e T)v_e}{s_e^2}\right) \\
s_e(v) &= \frac{s_0 + vT}{\sqrt{1 - \left(\frac{v}{v_0}\right)^4}}
\end{aligned}$$

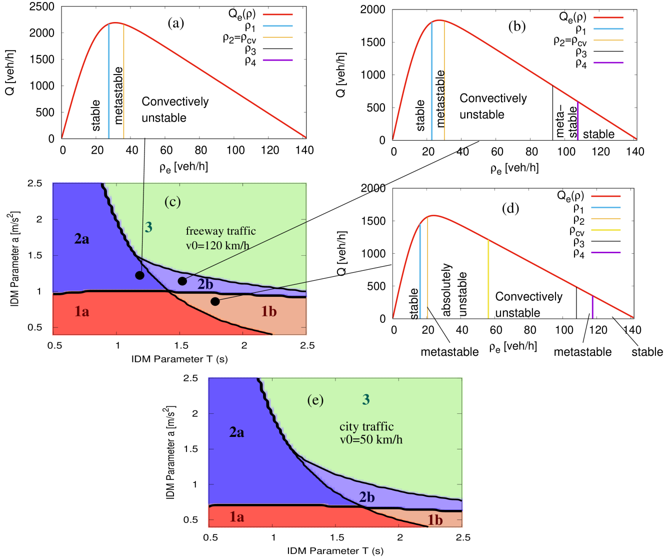


IDM instability condition (ctnd)

$$v_e' > \frac{1}{2}\left(f_l - f_v\right) = \frac{2a}{v_e}\left(\frac{v_e}{v_0}\right)^4 + \frac{s_0 + v_e T}{s_e^2}\left(aT + \sqrt{\frac{a}{b}}v_e\right)$$

- Stability increases with agility a and time gap T
- Stability also increases with decreasing b corresponding to an increased anticipation and decreasing v_0 and s_0
- Simple expression for near standstill, $v_e \to 0$, $s_e \to s_0$, $v_e'(s) \to 1/T$, so

$$\frac{1}{T} > \frac{s_0 a T}{s_e^2} \quad \text{or} \quad a < \frac{s_0}{T^2}$$



12.5 Flow Stability of Macroscopic Models

- General second-order macromodel including local and nonlocal terms:

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial x} &= D\frac{\partial^2 \rho}{\partial x^2}, \\ \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} &= A\left(\rho, V, \rho_a, V_a, \rho_x, V_x, \rho_{xx}, V_{xx}\right) \end{aligned}} \tag{4}$$

- Partial derivatives and nonlocalities:

$$\rho_x = \frac{\partial \rho(x,t)}{\partial x}, \quad \rho_{xx} = \frac{\partial^2 \rho(x,t)}{\partial x^2}, \quad \rho_a(x,t) = \rho(x_a, t) \text{ with } x_a > x.$$

- Partial speed derivatives V_x, V_{xx} and nonlocalities V_a defined in analogy

Steady state of the general macroscopic model

- Steady state: $A\left(\rho, V_e(\rho), \rho, V_e(\rho), 0, 0, 0, 0\right) = 0$
- Relation between $V_e'(\rho)$ and the acceleration derivatives:

$$\mathrm{d}A = \left(A_\rho + A_{\rho_a}\right)\,\mathrm{d}\rho + \left(A_V + A_{V_a}\right) V_e'(\rho)\,\mathrm{d}\rho = 0$$

$\Rightarrow$

$$\boxed{V_e'(\rho) = -\frac{A_\rho + A_{\rho_a}}{A_v + A_{V_a}}} \qquad (5)$$

- Acceleration sensitivities:

$$A_\rho = \left.\frac{\partial A}{\partial \rho}\right|_e, \quad A_{\rho_a} = \left.\frac{\partial A}{\partial \rho_a}\right|_e, \quad A_{\rho_x} = \left.\frac{\partial A}{\partial \rho_x}\right|_e, \quad A_{\rho_{xx}} = \left.\frac{\partial A}{\partial \rho_{xx}}\right|_e.$$

A_V, A_{V_a}, A_{V_x}. $A_{V_{xx}}$ in analogy

Linearisation of the general macromodel

Ansatz

$$
\begin{aligned}
\rho(x,t) &= \rho_e + \tilde{\rho}(x,t), \\
V(x,t) &= V_e + \tilde{V}(x,t),
\end{aligned}
$$

Linearisation of (4):

$$
\frac{\partial \tilde{\rho}}{\partial t} = -\rho_e \frac{\partial \tilde{V}}{\partial x} - V_e \frac{\partial \tilde{\rho}}{\partial x} + D \frac{\partial^2 \tilde{\rho}}{\partial x^2}, \tag{6}
$$

$$
\begin{aligned}
\frac{\partial \tilde{V}}{\partial t} = &-V_e \frac{\partial \tilde{V}}{\partial x} + A_\rho \tilde{\rho} + A_V \tilde{V} + A_{\rho_a} \tilde{\rho}_a + A_{V_a} \tilde{V}_a \\
&+ A_{\rho_x} \frac{\partial \tilde{\rho}}{\partial x} + A_{V_x} \frac{\partial \tilde{V}}{\partial x} + A_{\rho_{xx}} \frac{\partial^2 \tilde{\rho}}{\partial x^2} + A_{V_{xx}} \frac{\partial^2 \tilde{V}}{\partial x^2}
\end{aligned} \tag{7}
$$

shortcuts $\tilde{\rho}_a(x,t) = \tilde{\rho}(x_a,t)$ and $\tilde{V}_a(x,t) = \tilde{V}(x_a,t)$.

Linearisation: wave ansatz

Wave ansatz (Fourier modes) in the **Eulerian** instead of the **Lagrangian** frame of reference of the micromodels and in *physical* coordiates:

$$\begin{pmatrix} \tilde{\rho}_k(x,t) \\ \tilde{V}_k(x,t) \end{pmatrix} \propto \begin{pmatrix} \hat{\rho} \\ \hat{V} \end{pmatrix} \mathrm{e}^{\lambda t - \mathrm{i}kx} = \begin{pmatrix} \hat{\rho} \\ \hat{V} \end{pmatrix} \mathrm{e}^{(\sigma + \mathrm{i}\omega)t - \mathrm{i}kx}$$

- Complex growth rate $\lambda(k) = \sigma(k) + \mathrm{i}\omega(k)$ **Give the phase of the wave at (x,t)** $\phi = \omega(k)t - kx$
  Give the condition for stability All waves k have a real part of the growth rate $\sigma(k) = \mathrm{Re}\lambda(k) \leq 1$
- Physical wavelength $2\pi/k$, physical wavenumber k **Compare with the physical wavelength of micromodels** There, k is the phase shift from vehicle to vehicle with vehicle distance $s_e + l$, so physical wavelength $2\pi(s_e + l)/k$
- With I lanes, a wave contains $I\rho_e 2\pi/k$ vehicles. **How many vehicles does a single-lane car-following wave have?** $2\pi/k$ vehicles
- The points of constant phase $\phi = \omega t - kx$ (e.g., the wave crests) move with the velocity $\tilde{c}(k) = \omega/k$ in the stationary system. This has to be contrasted with the physical propagation velocity $\tilde{c}_{\text{mic}}(k) = v_e(s_e) + (s_e + l)\frac{\omega}{k}$ of microscopic waves in the stationary system.

Determining the waves

Wave ansatz into the linear system (6), (7) is only nontrivially solvable if the **eigenvalue** condition is satisfied:

$$\text{Det}\begin{pmatrix} \mathrm{i}kV_e - Dk^2 - \lambda & \mathrm{i}k\rho_e \\ A_\rho + A_{\rho_a}\mathrm{e}^{-\mathrm{i}ks_a} - \mathrm{i}kA_{\rho_x} - k^2A_{\rho_{xx}} & \mathrm{i}kV_e + A_V + A_{V_a}\mathrm{e}^{-\mathrm{i}ks_a} - \mathrm{i}kA_{V_x} - k^2A_{V_{xx}} - \lambda \end{pmatrix} \stackrel{!}{=} 0$$

Quadratic equation for the complex growth rate: $\lambda^2 + p(k)\lambda + q(k) = 0$ Solution:

$$\lambda_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} = -\frac{p}{2}\left(1 \pm \sqrt{1 - \frac{4q}{p^2}}\right)$$

with

$$p = p_0 + p_1 k + \mathcal{O}(k^2), \quad q = q_1 k + q_2 k^2 + \mathcal{O}(k^3)$$

and

$$\begin{aligned}
p_0 &= -(A_V + A_{V_a}), \\
p_1 &= \mathrm{i}(A_{V_x} + s_a A_{V_a} - 2V_e), \\
q_1 &= \mathrm{i}V_e(A_V + A_{V_a}) - \mathrm{i}\rho_e(A_\rho + A_{\rho_a}) = -\mathrm{i}Q'_e p_0, \\
q_2 &= V_e(A_{V_x} + s_a A_{V_a}) - \rho_e(A_{\rho_x} + s_a A_{\rho_a}) - V_\mathrm{e}^2 - D(A_V + A_{v_a})
\end{aligned}$$

Longwave stability criterion

Taylor approximation of $\lambda(k)$ around $k = 0$ (long-wavelength limit):

- Since $q = \mathcal{O}(k)$ while $p! = 0$ also for $k \to 0$, the square root can be expanded to second order in $\epsilon = 4q/p^2$ to ensure second order in k:

$$\sqrt{1-\epsilon} = 1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

- Since p_0 is real and $\lambda(k)$ should tend to zero for $k \to 0$ instead of tending to $-p_0$, the minus sign selects the more unstable mode in the long-wavelength limit:

$$\begin{aligned}
\lambda \quad &= \quad -\frac{p}{2}\left(\frac{\epsilon}{2} + \frac{\epsilon^2}{8}\right) + \mathcal{O}(\epsilon^3) \\
&\overset{q=\mathcal{O}(k)}{=} \quad -\left(\frac{q}{p} + \frac{q^2}{p^3}\right) + \mathcal{O}(k^3) \\
&= \quad -\left(\frac{q_1}{p_0}\right)k + \left(-\frac{q_2}{p_0} + \frac{q_1 p_1}{p_0^2} - \frac{q_1^2}{p_0^3}\right)k^2 + \mathcal{O}(k^3) \\
&\overset{q_1/p_0=-iQ'_e}{=} \quad iQ'_e(\rho_e)k + \left(\frac{-q_2 - ip_1 Q'_e(\rho_e) + (Q'_e(\rho_e))^2}{p_0}\right)k^2 + \mathcal{O}(k^3)
\end{aligned}$$

General result

- For $k \to 0$, $\lambda(k)$ tends to 0 which is alredy implied by vehicle number conservation
- The linear order in k is purely imaginary and determines the wave propagation with the wave velocity

$$\boxed{c = \frac{\omega}{k} = \frac{\mathrm{Im}\,\lambda}{k} = Q_e'(\rho_e)}$$

$\Rightarrow$ also the linear waves of second-order models obey the wave velocity formula of the LWR models (this is no longer the case for larger k!)

- The quadratic order is purely real and determines string stability. Since $p_0 = -(A_V + A_{V_a})$ is always < 0 for plausible models, we have the general longwavelength stability criterion for all local and nonlocal models

$$\boxed{(Q_e'(\rho_e))^2 - ip_1 Q_e'(\rho_e) - q_2 \leq 0} \tag{8}$$

Application to local and nonlocal models

- Local models ($A_{\rho_a} = 0$, $A_{V_a} = 0$) after replacing $Q'_e = \frac{\mathrm{d}}{\mathrm{d}\rho}(\rho V_e) = V_e + \rho V'_e$ (many terms cancel out!):

$$(\rho_e V'_e)^2 \leq -\rho_e \left(V'_e A_{V_x} + A_{\rho_x}\right) - D A_V \quad \text{Flow stability for local macroscopic models.} \tag{9}$$

- Special case that $A_{V_x} = 0$ and A_{ρ_x} can be written as a differential $-\frac{1}{\rho}\partial P/\partial x = -\frac{1}{\rho} P'(\rho_e)\frac{\partial \rho}{\partial x} = -\frac{1}{\rho} P'_e \frac{\partial \rho}{\partial x}$:

$$(\rho_e V'_e)^2 \leq P'_e - D A_V \tag{10}$$

- Nonlocal models (no gradients except for the $V\frac{\partial V}{\partial x}$ and pressure terms):

$$(\rho_e V'_e)^2 \leq P'_e - \rho_e s_a \left(V'_e A_{V_a} + A_{\rho_a}\right) \quad \text{Stability condition for nonlocal macro-models.} \tag{11}$$

Discussion

- In contrast to microscopic models, the speed sensitivity A_V alone does not influence stability since it appears only in combination with the density diffusion D which is zero, in most macroscopic models.

- Another surprising result: In spite of its obvious role to "smear out" gradients, the speed diffusion term $A_{V_{xx}}$ does *not* enter the stability criterion at all (while the density diffusion, if it exists, does)

- Most remarkable:

  Without gradients or nonlocalities, macroscopic models are unconditionally unstable: Anticipation is absolutely necessary

- Speed anticipations ($-\rho_e V_e' A_{V_x} \geq 0$ and $-\rho_e s_a V_e' A_{V_a} \geq 0$ for models fulfilling the acceleration plausibility criteria) increase the rhs of the criteria and therefore acts, as expected, stabilizing

- As expected, density anticipations ($-\rho_e A_{\rho_x} \geq 0$ and $-\rho_e s_a A_{\rho_a} \geq 0$) stabilize as well

Example 1: flow stability for Payne's model

Acceleration function $A(x,t) = \frac{V_e(\rho)-V}{\tau} + \frac{V_e'(\rho)}{2\rho\tau}\frac{\partial\rho}{\partial x}$

- We have $D=0$ and the acceleration sensitivities $A_{\rho_x} = V_e'/(2\rho\tau)$, and $A_{V_x}=0$
- Using (9), we obtain (watch out for the digns when multiplying both sides with $V_e' < 0$!)

$$\begin{aligned} \rho V_e'^2 &\le -A_{\rho_x} = -\frac{V_e'}{2\rho\tau} \\ -V_e'(\rho) = |V_e'(\rho)| &\le \frac{1}{2\rho^2\tau} \end{aligned}$$

Derive this using (10) and the pressure term

- ▷ The anticipation term $V_e'/(2\rho\tau)\frac{\partial\rho}{\partial x}$ can be written as the pressure gradient $-\frac{1}{\rho}P'(\rho)\frac{\partial\rho}{\partial x}$, so $P'(\rho) = -V_e'/(2\tau)$ and (10) reads $(\rho_e V_e')^2 \le -V_e'/(2\tau)$ resulting in the same stability condition

- As expected, stability increases with decreasing destabilizing force (speed-density sensitivity $|V_e'|$) and increasing agility (decreasing speed adaptation time τ). Specific to this model, stability also increases for very high densities

Example 2: flow stability for the GKT model

Acceleration function

$$
\begin{aligned}
A(x,t) &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{V_e^*(\rho, V, \rho_\mathrm{a}, V_\mathrm{a}) - V}{\tau}, \\
P(\rho) &= \rho\sigma_V^2(\rho) := \rho\alpha(\rho)V_e^2(\rho), \\
V_\mathrm{e}^*(\rho, V, \rho_\mathrm{a}, V_\mathrm{a}) &= V_0\left[1 - \frac{\alpha(\rho)}{\alpha(\rho_\mathrm{max})}\left(\frac{\rho_\mathrm{a} V T}{1-\rho_\mathrm{a}/\rho_\mathrm{max}}\right)^2 B\left(\frac{V - V_\mathrm{a}}{V\sqrt{2\alpha(\rho)}}\right)\right]
\end{aligned}
$$

- $\alpha(\rho)$: squared empirical speed variation coefficient σ_v/V_e,
- "Boltzmann factor" $B(x)$ with $B(0)=1$ and $B'(0) = 2\sqrt{2/\pi}$
- $V_a = V(x_a)$ with anticipation distance $s_a = x_a - x = \gamma(l_\mathrm{eff} + V_0 T)$ with $l_\mathrm{eff} = 1/\rho_\mathrm{max}$, anticipation factor γ
- Avoid numerical relaxation instabilities for ρ near ρ_max comming from the stiffness of the model for this situation: set (for a given update time Δt) $\tau \to (\rho) = \max(\tau, \Delta t(1 + 2V_0/V_e)$

- Nonlocal model with acceleration sensitivities (replace the $\alpha(\rho)/\alpha(\rho_\mathrm{max})$ terms with multiples of $(V_0 - V_e^*)$)

$$
\begin{aligned}
A_{\rho_a} &= -\frac{2(V_0 - V_e)\rho_\mathrm{max}}{\tau\rho_e(\rho_\mathrm{max} - \rho_e)} \\
A_{v_a} &= \frac{2(V_0 - V_e)}{\tau V_e\sqrt{\alpha\pi}}
\end{aligned}
$$

Flow stability for the GKT model (ctnd)

- Resulting stability criterion

$$(\rho_e V_e')^2 \leq P_e'(\rho) + \frac{2\gamma(l_{\text{eff}} + V_e T)(V_0 - V_e)}{\tau} \left[\frac{\rho_{\max}}{\rho_{\max} - \rho_e} - \frac{\rho_e V_e'}{V_e \sqrt{\alpha\pi}} \right]$$

- GKT stability...
  - increases with γ characterizing the level of anticipation,
  - increases with the driver's agility $1/\tau$,
  - increases with increasing desired time gap T, i.e., reducing the aggressiveness,
  - and increases with the sensitivity to speed differences which is characterized by $\alpha^{-1/2}$.
- Restabilisation limit for $\rho \to \rho_{\max}$ (with $V_e \approx \frac{1}{T}(\frac{1}{\rho} - \frac{1}{\rho_{\max}})$, $\rho V_e' \approx -\frac{1}{T\rho}$)

$$\gamma > \frac{\tau V_e}{2TV_0 \left[1 + (\alpha_{\max}\pi)^{-1/2}\right]}$$

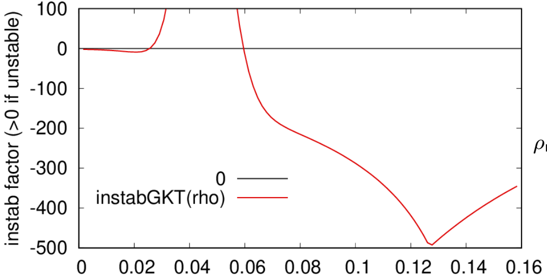
or with $\tau = \Delta t(1 + 2V_0/V_e) \approx 2\Delta t V_0/V_e$

$$\gamma > \frac{\Delta t}{T \left[1 + (\alpha_{\max}\pi)^{-1/2}\right]}$$

$\Rightarrow$ Restabilisation occurs for any sensible parameter set!

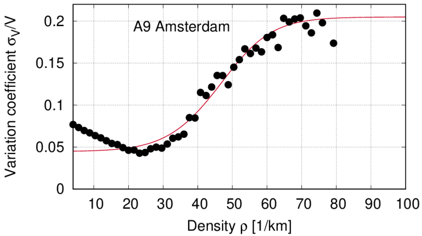
GKT instability

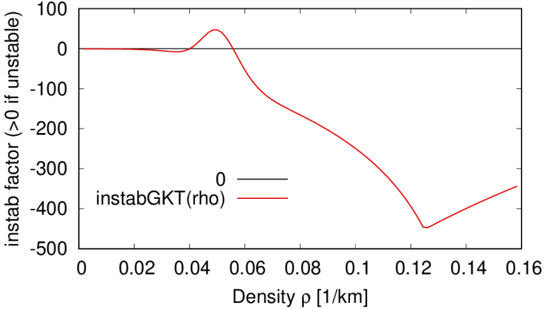
Instability indicator $$(\rho_e V_e')^2 - P_e'(\rho) - \frac{2\gamma(l_{\text{eff}}+V_e T)(V_0 - V_e)}{\tau}\left[\frac{\rho_{\text{max}}}{\rho_{\text{max}}-\rho_e} - \frac{\rho_e V_e'}{V_e\sqrt{\alpha\pi}}\right]$$



freeway

$$V_0 = 120\,\text{km/h}$$
$$T = 1.2\,\text{s}$$
$$\rho_{\text{max}} = 160\,/\text{km}$$
$$\tau_0 = 20\,\text{s}$$
$$\gamma = 1.2$$
$$\Delta t = 0.4\,\text{s}$$

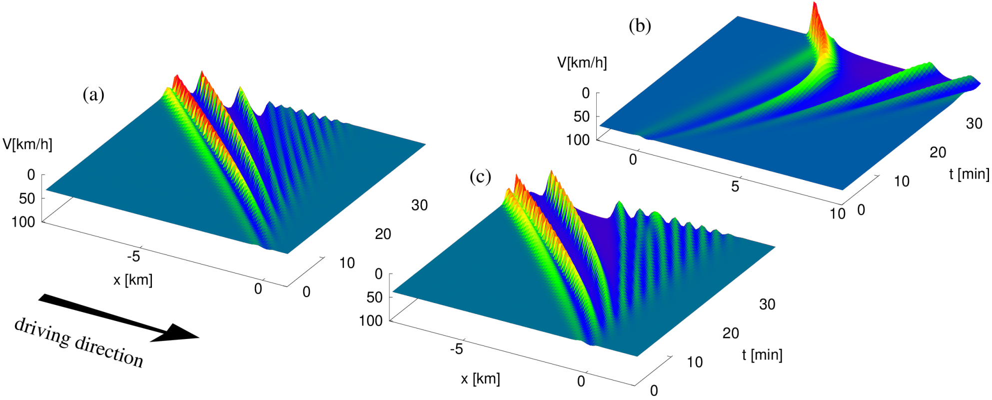




city

$$V_0 = 50\,\text{km/h}$$
$$\tau_0 = 8\,\text{s}$$

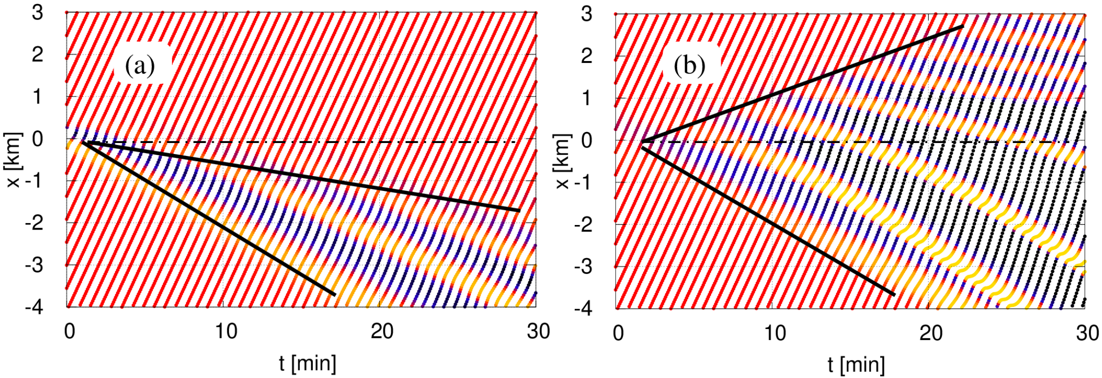
12.6 Convective Instability (only for Experts)



Convective (string) instability means that, while growing ($\text{Re}\lambda(k) > 0$ for some k or $|G(i\omega)| > 1$ for some ω), the waves are *convected away* after some time. After a localized initial perturbation $U(x, 0) = U_0(x)$, the amplitude $U(x, t)$ satisfies

- Maximum $\max_x U(x, t)$ grows over time after some transients (string instability)
- Amplitude $\lim_{t\to\infty} U(x, t) \to 0$ for $x \geq 0$ (**upstream convective instability**) or $x \leq 0$ (**downstream convective instability**). Directions for traffic flow? Mainly upstream but theoretically, donstream convective instability is possible as well for low densities and very unstable flows

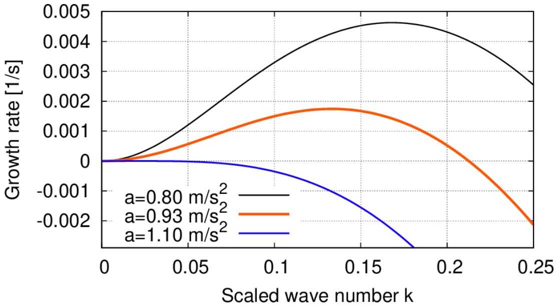
Convective instability for microscopic models



Convective instability is always defined in the stationary **Eulerian** reference frame for physical dimensions. For microscopic models in the linear regime, this means

- Physical wavenumber $k^{\text{phys}}(k) = \rho_e k = k/(l_{\text{veh}} + s_e)$
- Physical frequency at a constant location $\omega^{\text{phys}}(k) = v_e \rho_e k + \text{Im } \lambda(k)$ Recapitulate the meaning of the wavenumber k in micromodels in terms of the number of vehicles in a wave $2\pi/k$ vehicles in a wave, physical wavelength $(l_{\text{veh}} + s_e)2\pi/k$

Analytical approach



- Start from a homogeneous steady state (ρ_e, V_e) (without loss of generality macroscopic)
- Expand the **dispersion relation** (complex growthrate $\lambda(k)$) not around the wavenumber $k=0$ of the *first instability* but around the wavenumber k_0 of *maximum growthrate* $k_0 = \text{arg}(\text{max}_k \text{Re } \lambda(k))$ Why is, beyond the limit of string instability, the fastest growing mode never the first unstable mode $k \to 0$? Because it follows from vehicle conservation that waves with a wavelength $\to \infty$ can never grow nor shrink in amplitude, $\sigma \to 0$ for $k \to 0$
- Start with "perfectly" localized inititial perturbation (e.g., the speed perturbation)

$$U(x,0) = \delta(x), \quad \delta(x) = 0 \ \forall x \neq 0, \quad \int_{-\infty}^{\infty} \delta(x) = 1$$

Principle of the calculation

- The initial speed is Fourier-transformed in space. Since the Fourier transform of the δ-distribution is $=1$ for all wavenumbers k, we have complex Fourier modes (chose the *slow* mode with the higher growth rate Re λ)

$$\tilde{U}_k(x,t) = e^{\lambda(k)t - \mathrm{i}kx}$$

- An inverse Fourier transform (summing up over all the modes) gives the complex speed perturbation for any x and t

$$\tilde{U}(x,t) = \int_{k=-\infty}^{\infty} \tilde{U}_k(x,t)\, \mathrm{d}k$$

- In order to be analytically tractable, $\lambda(k)$ is expanded around k_0 giving complex Gaussian integrals which can be solved (but lengthy calculations)

Result

$$U(x,t) = \mathrm{Re}(\tilde{U}(x,t)) \tag{12}$$

$$\boxed{\tilde{U}(x,t) \propto \exp\left[\mathrm{i}(k_0^{\mathrm{phys}} x - \omega_0 t)\right] \, \exp\left[\left(\sigma_0 - \frac{\left(v_g - \frac{x}{t}\right)^2}{2(\mathrm{i}\omega_{kk} - \sigma_{kk})}\right) t\right]} \tag{13}$$

Quantity	Microscopic models	Macroscopic models
k_0^{phys}	$\rho_e k_0 = \rho_e \arg \max\limits_k \mathrm{Re}\, \lambda(k)$	$k_0 = \arg \max\limits_k \mathrm{Re}\, \lambda(k)$
σ_0	$\mathrm{Re}\ \lambda(k_0)$	$\mathrm{Re}\ \lambda(k_0)$
ω_0	$v_e \rho_e k_0 + \mathrm{Im}\ \ \lambda(k_0)$	$\mathrm{Im}\ \lambda(k_0)$
v_g	$v_e + \mathrm{Im}\ \lambda'(k_0)/\rho_e$	$\mathrm{Im}\ \lambda'(k_0)$
σ_{kk}	$\mathrm{Re}\ \lambda''(k_0)/\rho_e^2$	$\mathrm{Re}\ \lambda''(k_0),$
ω_{kk}	$\mathrm{Im}\ \lambda''(k_0)/\rho_e^2$	$\mathrm{Im}\ \lambda''(k_0).$

Signal velocities and the limits of the convective instability

- Evaluate the growth rate of (12) along the ray $x = ct$ corresponding to a *signal velocity* c:

$$\sigma(c) = \sigma_0 - \mathrm{Re}\left(\frac{(v_g - c)^2}{2(\mathrm{i}\omega_{kk} - \sigma_{kk})}\right) = \sigma_0 - \left(\frac{(v_g - c)^2}{2D_2}\right)$$

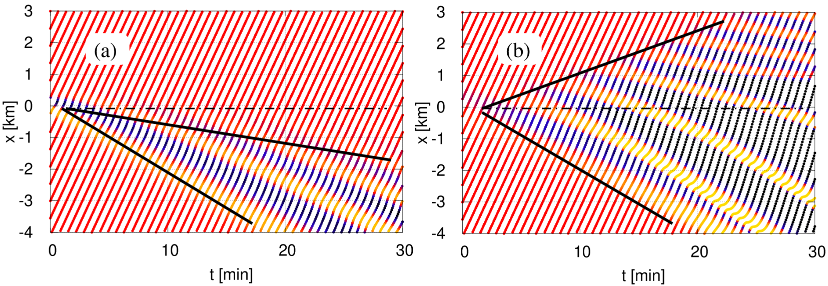
with

$$D_2 = -\sigma_{kk}\left(1 + \frac{\omega_{kk}^2}{\sigma_{kk}^2}\right)$$

How to calculate?

$1/D_2 = \mathrm{Re}(1/(\mathrm{i}\omega_k k - \sigma_{kk})) = -\sigma_{kk}/(\sigma_{kk}^2 + \omega_{kk}^2) = -1/(\sigma_{kk}(1 + \omega_{kk}^2/\sigma_{kk}^2))$

Signal velocities and the limits of the convective instability (ctnd)



- $U(x,t) = \text{Re}(\tilde{U}(x,t))$ grows in a range of rays bounded by the **signal velocities**

$$\sigma(c_s) \stackrel{\text{def}}{=} 0 \quad \Rightarrow \quad c_\text{s}^{\pm} = v_\text{g} \pm \sqrt{2D_2\sigma_0}$$

- At the limit between convective and absolute instability, one signal velocity is $=0$, so $\sigma_0 = \frac{v_g^2}{2D_2}$. The other limit is any string instability, so

$$\text{Convective instability} \quad \Longleftrightarrow \quad 0 < \sigma_0 \leq \frac{v_g^2}{2D_2}$$

"Reallity check" by simulation (IDM)